Introduction to univariate AR state-space models

Eli Holmes

FISH 507 – Applied Time Series Analysis

21 January 2019
Points from Thursday

- Data affected by a perturbation is problematic for arima(), Arima().
- Seasonal ARIMA has effect of Jan (or Feb ...) in year t on Jan (or Feb ...) in year t+1. Not typical when working with population data.
- Removing the mean season is different than a seasonal difference.
- Data with multiple seasons (daily, monthly, yearly) will be problematic for standard ARIMA seasonal models.
- Linear effects of past values might be problematic.
Weeks 1-3.5: building blocks for analysis of multivariate time-series data with observation error, structure, and missing values

- Matrix math (multivariate)
- Properties of time series data
- AR and MA models
  - State-space models: observation + process model
- Model evaluation and model selection
- Fitting models with STAN (non-linear, non-Gaussian, disparate data streams)

Starting next week: we will put this all together to start analyzing ecological data sets
The x model is the classic “random walk”. This model is a random walk observed with error.

\[
x_t = x_{t-1} + u + w_t, \quad w_t \sim \text{Normal}(0, q) \\
y_t = x_t + \nu_t, \quad \nu_t \sim \text{Normal}(0, r)
\]
univariate linear state-space model

\[ x_t = x_{t-1} + u + w_t, \quad w_t \sim Normal(0, q) \]

\[ y_t = x_t + v_t, \quad v_t \sim Normal(0, r) \]

Many textbooks on this class of model. Used extensively in economics and engineering.
Definition: AR-1 or AR lag-1

Value at time $t$ is the value at time $t-1$ plus random error

$$x_t = x_{t-1} + u + w_t$$

$$x_{t+1} = x_t + w_t$$

$$x_t = bx_{t-1} + u + w_t$$
Addition of “b” (<1) leads to process model with mean-reversion,

\[ N_t = \exp(u + e_t)N_{t-1}^b \]

\[ x_t = bx_{t-1} + u + e_t \]

\[ e_t \sim \text{Normal}(0, q) \]

Log-space

Weak density-dependence

Strong density-dependence

b<1: Gompertz density-dependent process
This model is quite hard to fit

\[ N_t = \exp(u + e_t)N_{t-1}^b \]

\[ x_t = bx_{t-1} + u + e_t \]

\[ e_t \sim \text{Normal}(0, q) \]

b and u are confounded = ridge likelihood = many b/u combinations that fit the data

If you have observation error, you need either long times or replication to estimate this model.
Why is the AR-1 model so important in analysis of ecological data?

Additive random walks
• Movement, changes in gene frequency, somatic growth if growth is by fixed amounts

\[ x_t = x_{t-1} + u + w_t, \quad w_t \sim Normal(0, q) \]

Multiplicative random walks
• Population growth, somatic growth if growth is by percentage

\[ n_t = \lambda n_{t-1} w_t, \quad w_t \sim \log-Normal(0, q) \]

• take log and you get the linear additive model above. log-normal means that 10% increase is as likely as 10% decrease
An AR-1 random walk can show a wide-range of trajectories, even for the same parameter values.

All trajectories came from the same rw model: \( x_t = x_{t-1} - 0.02 + e_t, \) \( e_t \sim \text{Normal}(\text{mean}=0.0, \text{var}=0.01) \)

same as the “stochastic exponential growth model”: \( N_t = N_{t-1} \exp(-0.02+e_t) \)
Definition: state-space

The “state”, the x, is a hidden (dynamical) variable. In this class, it is a **hidden random walk**.

Our data, y, are observations of this.

*Often state-space models include inputs (explanatory variables). and typically at least the x is multivariate, and often also y.*

The model you are seeing today is a simple univariate state-space model with no inputs.

*state process* \[ x_t = x_{t-1} + u + w_t, \quad w_t \sim Normal(0, q) \]

*obs process* \[ y_t = x_t + v_t, \quad v_t \sim Normal(0, r) \]
univariate example: population count data

Yearly (usually) population (or subpopulation) counts

Pop. Estimate of Monk Seals

Year

Pop Estimate


Missing values
Observation error

There IS some number of sea lions in our population in year \( x \), but we don’t know that number precisely. It is “hidden”.
Suppose we have the following data (population counts logged)
A linear regression model

\[ y_t = \alpha_0 + \beta t + z_t; z_t \sim Normal(0, \sigma) \]

Regression is fitting a deterministic process through the data
no “process” variability
all variability = “non-process or observation error”
Autoregressive state-space models fit a RANDOM WALK through the data

variability = “observation error” + “process error”
Two types of variability

#1 observation or “non-process” variability

Difference between observation and process is the non-process error

log(N)

t
Two types of variability

#1 observation or “non-process” variability

The non-process (observation) variance is often unknowable in fisheries and ecological data

- **Sightability varies due to factors that may not be fully understood or measureable**
  - Environmental factors (tides, temperature, etc.)
  - Population factors (age structure, sex ratio, etc.)
  - Species interactions (prey distribution, prey density, predator distribution or density, etc.)

- **Sampling variability**—due to how you actually count animals—is just one component of observation variance
This the process line. It ‘wiggles’ due to process variability. The next few lectures we will focus on processes that are simple random walks with drift: \( x_t = x_{t-1} + u + w_t, \quad w_t \sim \text{Normal}(0, q) \)
Process error is the difference between the expected $x(t)$ and the actual value.

Let’s say that in $x(t)=x(t-1)-0.02+e(t)$.*

The difference between “$x$”, the expected value, and the blue square, actual value, is the process error.

“$x$” shows the expected $x(t)$ at time $t$; it is like the prediction from a deterministic process.

*If this were a population model, that means a the mean rate of decline is ca 2% per year.
One use of univariate state-space models is “count-based” population viability analysis (chap 6 HWS2014)
How you model your data has a large impact on your forecasts.
How can we separate process and non-process variance?

Wouldn’t these two variances be confounded and impossible to estimate simultaneously?
How can we separate process and observation variance?
They have different temporal patterns.

**Process error:** $x_t = x_{t-1} + u + e_t$

multiple sims of $x(t)$ with same $u$ and $q$

**Observation error:** $y_t = x_t + \eta_t$

multiple sims of $y(t)$ with same $x(t)$
An AR-1 state-space model combines a model for the hidden AR-1 process with a model for the observation process...

...and allows us to separate the variances.

**Process model**

\[ x_t = x_{t-1} + u + w_t \]

\[ w_t \sim \text{Normal}(0, q) \]

**Observation model**

\[ y_t = x_t + v_t \]

\[ v_t \sim \text{Normal}(0, r) \]

AR lag-1 random walk with drift
normally distributed process errors

observation errors
normally distributed process errors
A mathematical algorithm that solves for the ‘optimal’ (least error or maximum-likelihood) $x_t$ given all the data $(y)$ from time 1 to $t$.
Let’s simulate and try fitting some models

• Open up R and follow after me
• univariate_example_1.R
• univariate_example_2.R
• univariate_example_3.R
How to write a straight-line as AR-1

• ##Preliminaries: how to write
  ##x=\text{intercept}+\text{slope}*t\text{ as a AR-1}
• x(0)=\text{intercept}
• x(1)=x(0)+\text{slope} \text{ #this is } x\text{ at } t=1
• x(2)=x[1]+\text{slope}
• so..
• x(t)=x(t-1)+\text{slope}+w(t)\text{, }w(t)\sim\text{N}(0,0)
MARSS R Package

• Fits MARSS models (multivariate AR-1 state-space)
• General, fits any MARSS model with Gaussian errors

• But
• Maximum likelihood
• Slow. Students working with large data sets have gotten huge speed improvements by coding their models in TMB
MARSS R Package

• Fits MARSS models (multivariate AR-1 state-space)

• MARSS model syntax

\[ X(t) = B \ X(t-1) + U + w(t), \ w(t) \sim N(0, Q) \]
\[ Y(t) = Z \ X(t) + A + v(t), \ v(t) \sim N(0,R) \]

• \texttt{fit2=MARSS(y,model=mod.list)}

• \texttt{y} is data; \texttt{model} tells MARSS what the parameters are
• The parameters are MATRICES
• You write matrices just like they appear in your model on paper
• You pass \texttt{model} to MARSS as a list
MARSS model in matrix form

\[
\begin{bmatrix}
    y_{1,t} \\
y_{2,t} \\
y_{3,t} \\
y_{4,t} \\
y_{5,t}
\end{bmatrix}
= \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    0 & 0 & 1
\end{bmatrix}
\times \begin{bmatrix}
    x_{JF,t} \\
x_{N,t} \\
x_{S,t}
\end{bmatrix}
+ \begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix}
+ \begin{bmatrix}
\eta_{1,t} \\
\eta_{2,t} \\
\eta_{3,t} \\
\eta_{4,t} \\
\eta_{5,t}
\end{bmatrix}
\]
\[ X(t) = B X(t-1) + U + w(t), \ w(t) \sim N(0, Q) \]
\[ Y(t) = Z X(t) + A + v(t), \ v(t) \sim N(0, R) \]

mod.list=list(
    U=matrix("u"),
    x0=matrix(0),
    B=matrix(1),
    Q=matrix(0.1),
    Z=matrix(1),
    A=matrix(0),
    R=matrix("r"),
    tinitx=0)

Let's say we want to fit this model:

\[ x_t = x_{t-1} + u + w_t, \ w_t \sim N(0, \sigma^2 = 0.1) \]
\[ y_t = x_t + v_t, \ v_t \sim N(0, r) \]
\[ x_0 = 0 \]
Let’s simulate and try fitting some models

- Open up R and follow after me
- univariate_example_1.R
- univariate_example_2.R
- univariate_example_3.R
State-space diagnostics
Basic diagnostics

Nile River models from the lab handout
Basic diagnostics: #1 plot the residuals

Model residuals

State residuals

There should be no temporal trends!

They should be centered about 0.
non-process error or model residual

Difference between observation and process is the non-process error also called “model residual”
process error or state residual

Difference between the forecasted state at time $t$ given the state at time $t-1$ and the actual state at time $t$
Basic diagnostics: #2 check acf of residuals

v(t) are model residuals

w(t) are state residuals
Even our ‘best’ model is missing something...
Basic diagnostics: #3 Simulate from your estimated model and compare to the data.

Black line is the estimated state from model 2
Basic diagnostics #4 Simulate from known model and then test whether you can re-capture the true estimates
How do you know when to use a process error or observation error model?

• If your time-series data contain both types, use a model with both types.

• To estimate both variances, you need a) 20+ time steps OR b) multi-site data.

• If you don’t have enough data, you need to use assumptions about one of the variances. Meaning a) fix the value or b) incorporate a prior.

• Diagnostics: Observation error induces autocorrelation in the noise of an autoregressive process. Fit a process-error only model (R=0) and check for autocorrelation of residuals.
Other types of “non-process” error

• Fluctuations that don’t have “feedback” (variance doesn’t explode)

• Lots of biological processes also create noise that looks like that
  – age-structure cycles  o cyclic variability in fecundity
  – density-dependence  o predator-prey interactions

• If your model cannot accommodate that cycling,
  – it tends to get ‘soaked’ up in the ‘non-process’ error component

• If your model can accommodate that cycling,
  – estimation of ‘observation error’ variance can be confounded, unless you have long, long datasets or replicates
Thursday lecture: multivariate state-space

\[
\begin{bmatrix}
    y_{1,t} \\
    y_{2,t} \\
    y_{3,t} \\
    y_{4,t} \\
    y_{5,t}
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    0 & 0 & 1
\end{bmatrix}
\times
\begin{bmatrix}
    x_{JF,t} \\
    x_{N,t} \\
    x_{S,t}
\end{bmatrix} +
\begin{bmatrix}
    a_1 \\
    a_2 \\
    a_3 \\
    a_4 \\
    a_5
\end{bmatrix} +
\begin{bmatrix}
    \eta_{1,t} \\
    \eta_{2,t} \\
    \eta_{3,t} \\
    \eta_{4,t} \\
    \eta_{5,t}
\end{bmatrix}
\]

Thursday lab: fitting univariate and multivariate state-space models